Capítulo 14

Lie Algebras and Financial Models

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Abstract: We show that Lie algebras can be used to study financial models. In particular we show that the Black-Scholes equation can be analyzed using the Schrödinger algebra.

Keywords: Lie algebra, Black-Scholes equation, Schrödinger equation.

Resumen: En este trabajo se muestra que las álgebras de Lie se pueden emplear para estudiar modelos financieros. En particular se muestra que las simetrías de la ecuación de Schrödinger se pueden usar para estudiar las simetrías de la ecuación de Black-Scholes que se usa para determinar el precio de opciones financieras.

Palabras clave: Algebras de Lie, Ecuación de Schrödinger, Ecuación de Black-Scholes.

14.1 Introduction

The Black-Scholes equation is the most important equation in finance theory [Black1973, Merton1973]. In fact, this equation is one of the most beautiful and useful in applied mathematics. Amazingly, this equation was constructed taking simple assumptions, for example that the prices follow a normal distribution.

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However, there are some realistic situations where the Black-Scholes equation assumptions are not satisfied. For example, in the Black-Scholes equation the arbitrage is not considered and in the real financial world there is arbitrage. In addition, there are some markets where the prices fluctuations do not follow a normal distribution. If we use the Black-Scholes equation when its assumptions are not satisfy, we are mistaken and in a big problem. In fact, before the Black-Scholes equation was formulated, Mandelbrot noticed that some prices do not follow a normal distribution [Mandelbrot1977]. For that reason Mandelbrot proposed the Cauchy distribution to describe fluctuation prices. Recently, in orden to obtain a more realistic Black-Scholes equation, some authors haven been proposed different generalized Black-Scholes equations. For instance using, a stochastic volatility [Heston1993], multifractal volatility [Calvent2008], jump processes [Tankov2003], Levy's distributions [Carr2003], fractional differential equations [Kleinert20015], etc.

Notably mathematical techniques developed in physics have been employed to study financial phenomena [Bouchaud2003, Baaquie2004]. For example, the Black-Scholes equation can be mapped to the one dimensional free Schrödinger equation [Baaquie2004]. This mapping allowed the birth of a new discipline, the so call Quantum Finance [Baaquie2004]. Now, it is well known that when symmetries are present in a physical system we can get at the properties of a system without completely solving all the equations that describe the system, in fact symmetries imply conserved quantities. It is worth pointing out that the conformal symmetry is important in physics, for example the relativistic conformal group is the largest symmetry group of special relativity [Maldacena2000]. In addition, the free Schrödinger equation is invariant under the Schrödinger group, which is a non-relativistic conformal group [Hagen1972, Niederer1972]. Notably in 1882 Sophus Lie showed that the Fick equation, which describes diffusion, is invariant under the Schrödinger group [Lie1882].

Then, symmetry methods may help us understand financial phenomena. For that reason, in this work we study the Black-Scholes equation symmetries, in particular we show that this equation is invariant under the Schrödinger group. In order to do this, the one dimensional free non-relativistic particle and its symmetries will be revisited. To get the Black-Scholes equation symmetries, the particle mass is identified as the inverse of square of the volatility. Furthermore, using financial variables, a Schrödinger algebra representation is constructed. In addition, it is shown that the operators of this last representation are not hermitian and not conserved. This work is based in the paper [Romero2014].

This paper is organized as follows: in section 14.2 we provide a brief overview of the one dimensional non-relativistic free particle and its symmetries; in section

14.3 the one dimensional free Schrödinger equation and its symmetries are studied; in section 14.4 the Black-Scholes equation and its symmetries are studied. Finally, in section 14.5 a summary is given.

14.2 Free particle

The action for one-dimensional non-relativistic free particle is given by

$$S = \int dt \frac{m}{2} \left(\frac{dx}{dt} \right)^2 , \tag{14.2.1}$$

this is the simplest mechanical system. Now, using the coordinates transformation

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad x' = \frac{ax + vt + c}{\gamma t + \delta}, \qquad a^2 = \alpha\delta - \beta\gamma \neq 0, \tag{14.2.2}$$

where $\alpha, \beta, \gamma, \delta, a, v, c$ are constants, the action (14.2.1) transforms as

$$S' = \int dt' \frac{m}{2} \left(\frac{dx'}{dt'} \right)^2 = S + \frac{m}{2} \int dt \left(\frac{d\phi(x,t)}{dt} \right), \tag{14.2.3}$$

where

$$\phi(x,t) = \frac{1}{a^2} \left(2avx + v^2 t - \frac{\gamma \left(ax + vt + c \right)^2}{\gamma t + \delta} \right). \tag{14.2.4}$$

Then, the equation of motion for one dimensional free particle is invariant under the conformal coordinate transformations (14.2.2). This coordinates transformation includes temporal translations

$$t' = t + \beta, \qquad x' = x, \tag{14.2.5}$$

spatial translations

$$t', \qquad x' = x + c, \tag{14.2.6}$$

Galileo's transformations

$$t', \qquad x' = x + vt, \tag{14.2.7}$$

anisotropic scaling

$$t' = a^2 t, \qquad x' = ax \tag{14.2.8}$$

and the special conformal transformations

$$t' = \frac{1}{\gamma t + 1}, \qquad x' = \frac{x}{\gamma t + 1}. \tag{14.2.9}$$

It is shown below that the conformal coordinate transformations (14.2.2) and the quantity (14.2.4) are useful to study Black-Scholes equation symmetries.

14.2.1 Conservative quantities

For the one dimensional non-relativistic particle, the following quantities

$$P = m\dot{x}, \tag{14.2.10}$$

$$H = \frac{P^2}{2m}, \tag{14.2.11}$$

$$G = tP - mx, \tag{14.2.12}$$

$$K_1 = tH - \frac{1}{2}xP, \tag{14.2.13}$$

$$K_2 = t^2H - txP + \frac{m}{2}x^2 \tag{14.2.14}$$

are conserved.

The momentum P is associated with spatial translations (14.2.6). The Hamiltonian H is associated with temporal translations (14.2.5) . The quantity G is associated with Galileo‘s transformations (14.2.7). While K_1 is associated with anisotropic scaling (14.2.8) and K_2 is associated with the special conformal transformations (14.2.9).

Furthermore, using the Poisson brackets, it can be shown that the following relations

$$\{P, H\} = 0, \tag{14.2.15}$$

$$\{P, K_1\} = \frac{1}{2}P, \tag{14.2.16}$$

$$\{P, K_2\} = G, \tag{14.2.17}$$

$$\{P, G\} = m, \tag{14.2.18}$$

$$\{H, K_1\} = H, \tag{14.2.19}$$

$$\{H, G\} = P, \tag{14.2.20}$$

$$\{H, K_2\} = 2K_1, \tag{14.2.21}$$

$$\{K_1, K_2\} = K_2, \tag{14.2.22}$$

$$\{K_1, G\} = \frac{1}{2}G, \tag{14.2.23}$$

$$\{K_2, G\} = 0 \tag{14.2.24}$$

are satisfied.

14.2.2 Schrödinger group

In quantum mechanics, if a particle is in x_0 at time t_0, the amplitude to travel to x in a time $T = t - t_0$ is given by

$$U(x,t;x_0,t_0) = \left(\frac{m}{2\pi i(t-t_0)}\right)^{\frac{1}{2}} e^{\frac{i}{\hbar}\int_{t_0}^{t} d\tilde{t}\frac{m}{2}\left(\frac{dx}{d\tilde{t}}\right)^2}, \tag{14.2.25}$$

which satisfies the Schrödinger equation

$$i\hbar\frac{\partial U(x,t;x_0,t_0)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 U(x,t;x_0,t_0)}{\partial x^2}. \tag{14.2.26}$$

Now, in another system with coordinates x', t', x'_0, t'_0 we have the amplitude

$$U'(x',t';x'_0,t'_0) = \left(\frac{m}{2\pi i(t'-t'_0)}\right)^{\frac{1}{2}} e^{\frac{i}{\hbar}\int_{t'_0}^{t'} d\tilde{t}'\frac{m}{2}\left(\frac{dx'}{d\tilde{t}'}\right)^2}, \tag{14.2.27}$$

which satisfies the Schrödinger equation (14.2.26) on primed coordinates (x', t'). In addition, using the conformal transformations (14.2.2) we have

$$t' = \frac{\alpha t+\beta}{\gamma t+\delta}, \qquad x' = \frac{ax+vt+c}{\gamma t+\delta}, \tag{14.2.28}$$

$$t'_0 = \frac{\alpha t_0+\beta}{\gamma t_0+\delta}, \qquad x'_0 = \frac{ax_0+vt_0+c}{\gamma t_0+\delta}. \tag{14.2.29}$$

Notice that, without loss of generality, we can take $x_0 = 0$ and $t_0 = 0$, in this case

$$t' - t'_0 = \frac{a^2 t}{\delta(\gamma t+\delta)}. \tag{14.2.30}$$

Then, using the equation (14.2.3), we have

$$U'(x',t';x'_0,t'_0) = \sqrt{\frac{\delta}{a^2}}\left(\sqrt{\gamma t+\delta}\right) e^{\frac{im}{2\hbar}\phi(x,t)} U(x,t;0,0), \tag{14.2.31}$$

where $\phi(x,t)$ is given by (14.2.4). Now, due that $U(x,t;x_0,t_0)$ is solution for the Schrödinger equation (14.2.26) on coordinates (x,t), and $U'(x',t';x'_0,t'_0)$ satisfies the same equation on primed coordinates (x',t'), the expression (14.2.31) implies that the Schrödinger equation for the one dimensional non-relativistic free particle

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} \tag{14.2.32}$$

is invariant under the conformal coordinate transformations (14.2.2), where the wave function transforms as

$$\psi'(x',t') = \left(\sqrt{\gamma t+\delta}\right) e^{\frac{im}{2\hbar}\phi(x,t)}\psi(x,t), \tag{14.2.33}$$

and $\phi(x,t)$ is given by (14.2.4).

Using other methods, the conformal symmetry for the free Schrödinger equation was found by Niederer and Hagen in 1972 [Niederer1972, Hagen1972]. However, this symmetry was obtained by S. Lie in 1882 while he was studying the Fick equation [Lie1882].

Furthermore, according to quantum mechanics, the quantities (14.2.10)-(14.2.14) are represented by the operators

$$\hat{P} = -i\hbar\frac{\partial}{\partial x}, \tag{14.2.34}$$

$$\hat{H} = \frac{\hat{P}^2}{2m}, \tag{14.2.35}$$

$$\hat{G} = t\hat{P} - mx, \tag{14.2.36}$$

$$\hat{K}_1 = t\hat{H} - \frac{1}{4}\left(x\hat{P} + \hat{P}x\right), \tag{14.2.37}$$

$$\hat{K}_2 = t^2\hat{H} - \frac{t}{2}\left(x\hat{P} + \hat{P}x\right) + \frac{m}{2}x^2. \tag{14.2.38}$$

These operators satisfy the Schrödinger algebra

$$\left[\hat{P}, \hat{H}\right] = 0, \tag{14.2.39}$$

$$\left[\hat{P}, \hat{K}_1\right] = \frac{i\hbar}{2}\hat{P}, \tag{14.2.40}$$

$$\left[\hat{P}, \hat{K}_2\right] = i\hbar\hat{G}, \tag{14.2.41}$$

$$\left[\hat{P}, \hat{G}\right] = i\hbar m, \tag{14.2.42}$$

$$\left[\hat{H}, \hat{K}_1\right] = i\hbar\hat{H}, \tag{14.2.43}$$

$$\left[\hat{H}, \hat{G}\right] = i\hbar\hat{P}, \tag{14.2.44}$$

$$\left[\hat{H}, \hat{K}_2\right] = 2i\hbar\hat{K}_1, \tag{14.2.45}$$

$$\left[\hat{K}_1, \hat{K}_2\right] = i\hbar\hat{K}_2, \tag{14.2.46}$$

$$\left[\hat{K}_1, \hat{G}\right] = \frac{i\hbar}{2}\hat{G}, \tag{14.2.47}$$

$$\left[\hat{K}_2, \hat{G}\right] = 0, \tag{14.2.48}$$

which is similar to the algebra (14.2.15)-(14.2.24). It is possible to show that the operators (14.2.34)-(14.2.38) are conserved.

In the next section, it will be shown that Black-Scholes equation is invariant under Schrödinger symmetry.

14.3 Black-Scholes and Schrödinger equations

An essential question in financial world refers to how obtain the price of an option. Financial markets use the Black-Scholes option pricing model and it is the basis of sophisticated methods of options valuation. A remarkable result in this field is given by the Black-Scholes equation [Black1973, Merton1973]

$$\frac{\partial C(s,t)}{\partial t} = -\frac{\sigma^2}{2}s^2\frac{\partial^2 C(s,t)}{\partial s^2} - rs\frac{\partial C(s,t)}{\partial s} + rC(s,t), \tag{14.3.1}$$

where C is the price of an option, s is the price of the stock, σ is the volatility and r is the annualized risk-free interest rate. The Black-Scholes equation have to be solved with condition

$$C(s,T) = (s-K)\theta(s-K), \tag{14.3.2}$$

where K is the strike price of an option, T is the time to expiration and $\theta(x)$ is the Heaviside function.

Using the change of variable

$$s = e^x \tag{14.3.3}$$

in the equation (14.3.1), the following result

$$\frac{\partial C(x,t)}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial^2 C(x,t)}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\partial C(x,t)}{\partial x} + rC(x,t). \tag{14.3.4}$$

is gotten. Additionally, if

$$C(x,t) = e^{\left[\frac{1}{\sigma^2}\left(\frac{\sigma^2}{2}-r\right)x + \frac{1}{2\sigma^2}\left(\frac{\sigma^2}{2}+r\right)^2 t\right]}\Psi(x,t) \tag{14.3.5}$$

the following equation

$$\frac{\partial \Psi(x,t)}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial^2 \Psi(x,t)}{\partial x^2} \tag{14.3.6}$$

is obtained, which is Schrödinger-like wave equation (14.2.32). Notice that $1/\sigma^2$ has the role of particle mass m. In addition, with the change of variable $\tau = T - t$, we get

$$\frac{\partial \Psi(x,\tau)}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 \Psi(x,\tau)}{\partial x^2}, \tag{14.3.7}$$

which is the heat equation.

Now, using the solution for the free Schrödinger equation, we can get the solution for the Black-Sholes equation. In fact, using the initial condition (14.3.2) we get

$$C(s,t) = sN(d_+) - Ke^{-r(T-t)}N(d_-) \tag{14.3.8}$$

where

$$N(z) = \int_{-\infty}^{z} du \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}, \quad d_\pm = \frac{\ln\left(\frac{s}{K}\right) + (T-t)\left(r \pm \frac{\sigma^2}{2}\right)}{\sigma\sqrt{T-t}}.$$

The equation (14.3.8) is the so call Black-Scholes formula and it can be obtained using quantum mechanics techniques.

14.4 The Schrödinger group and the Black-Scholes equation

Due that the Schrödinger equation (14.2.32) is invariant under conformal transformation (14.2.2), the equation (14.3.7) is invariant under the same transformation. In this case the function $\psi(x,t)$ transforms as

$$\Psi'\left(x',t'\right) = \left(\sqrt{\gamma t+\delta}\right) e^{\frac{1}{2\sigma^2}\phi(x,t)}\Psi(x,t), \tag{14.4.1}$$

where $\phi(x,t)$ is given by (14.2.4). Notice that the particle mass m is changed for $1/\sigma^2$.

Using the change of variable (14.3.3), the coordinate transformations (14.2.2) can be written as

$$t' = \frac{\alpha t+\beta}{\gamma t+\delta}, \qquad s' = e^{\left(\frac{vt+c}{\gamma t+\delta}\right)} s^{\left(\frac{a}{\gamma t+\delta}\right)}. \tag{14.4.2}$$

Through a long but straightforward calculation, it can be shown that the Black-Scholes equation (14.3.1) is invariant under this last transformation, where the price $C(s,t)$ transforms as

$$C'\left(s',t'\right) = \left(\sqrt{\gamma t+\delta}\right) s^{\Phi_1(s,t)} e^{\Phi_2(s,t)} C(s,t), \tag{14.4.3}$$

here

$$\Phi_1(s,t) = \frac{-2a^2\gamma\left(\frac{\sigma^2}{2}-r\right)t + 2a\left(v\delta-\gamma c\right) + 2a^2(a-\delta)\left(\frac{\sigma^2}{2}-r\right)}{2a^2\sigma^2\left(\gamma t+\delta\right)} - \frac{\gamma a^2\left(\ln s\right)}{2a^2\sigma^2\left(\gamma t+\delta\right)}$$

and

$$\Phi_2(s,t) = \left(\frac{a^2\left(\frac{\sigma^2}{2}+r\right)^2(\alpha-\delta)+2a^2v\left(\frac{\sigma^2}{2}-r\right)+v\left(v\delta-2\gamma c\right)}{2\sigma^2a^2(\gamma t+\delta)} \right) t + \frac{a^2\beta\left(\frac{\sigma^2}{2}+r\right)^2+2a^2\left(\frac{\sigma^2}{2}-r\right)c-\gamma c^2-\gamma a^2\left(\frac{\sigma^2}{2}+r\right)^2 t^2}{2\sigma^2a^2(\gamma t+\delta)}.$$

Now, the Black-Scholes equation (14.3.1) can be written as

$$\frac{\partial C(s,t)}{\partial t} = \hat{\mathbf{H}}C(s,t), \tag{14.4.4}$$

where

$$\hat{\mathbf{H}} = -\frac{\sigma^2}{2}s^2\frac{\partial^2}{\partial s^2} - rs\frac{\partial}{\partial s} + r. \tag{14.4.5}$$

Moreover, using the operator

$$\hat{\Pi} = -is\frac{\partial}{\partial s} + \frac{i}{\sigma^2}\left(\frac{\sigma^2}{2}-r\right), \tag{14.4.6}$$

the operator $\hat{\mathbf{H}}$ can be rewritten as

$$\hat{\mathbf{H}} = \frac{\sigma^2}{2}\hat{\Pi}^2 + \frac{1}{2\sigma^2}\left(\frac{\sigma^2}{2}+r\right)^2. \tag{14.4.7}$$

Notice that the operator $\hat{\mathbf{H}}$ is similar to the Hamiltonian operator $\hat{H}$ (14.2.35), where the particle mass m is associated with $1/\sigma^2$. However, the operator $\hat{\mathbf{H}}$ is not hermitian.

Additionally, using the operator (14.4.6) it is possible construct quantities related with the non-relativistic free particle conserved quantities (14.2.34)-(14.2.38). In fact, the operators

$$\hat{\Pi} = -is\frac{\partial}{\partial s} + \frac{i}{\sigma^2}\left(\frac{\sigma^2}{2}-r\right), \tag{14.4.8}$$

$$\hat{\mathbf{H}}_0 = \frac{\sigma^2}{2}\hat{\Pi}^2, \tag{14.4.9}$$

$$\hat{\mathbf{G}} = t\hat{\Pi} - \frac{1}{\sigma^2}\ln s, \tag{14.4.10}$$

$$\hat{\mathbf{K}}_1 = t\hat{\mathbf{H}}_0 - \frac{1}{4}\left(\ln s\hat{\Pi} + \hat{\Pi}\ln s\right), \tag{14.4.11}$$

$$\hat{\mathbf{K}}_2 = t^2\hat{\mathbf{H}}_0 - \frac{t}{2}\left(\ln s\hat{\Pi} + \hat{\Pi}\ln s\right) + \frac{1}{2\sigma^2}(\ln s)^2 \tag{14.4.12}$$

can be constructed, which are similar to the quantities (14.2.34)-(14.2.38). Now, using the relation

$$[\ln s, \hat{\Pi}] = i, \tag{14.4.13}$$

the algebra

$$\left[\hat{\Pi}, \hat{\mathbf{H}}_0\right] = 0, \tag{14.4.14}$$

$$\left[\hat{\Pi}, \hat{\mathbf{K}}_1\right] = \frac{i}{2}\hat{\Pi}, \tag{14.4.15}$$

$$\left[\hat{\Pi}, \hat{\mathbf{K}}_2\right] = i\hat{\mathbf{G}}, \tag{14.4.16}$$

$$\left[\hat{\Pi}, \hat{\mathbf{G}}\right] = \frac{i}{\sigma^2}, \tag{14.4.17}$$

$$\left[\hat{\mathbf{H}}_0, \hat{\mathbf{K}}_1\right] = i\hat{\mathbf{H}}_0 \tag{14.4.18}$$

$$\left[\hat{\mathbf{H}}_0, \hat{\mathbf{G}}\right] = i\hat{\Pi}, \tag{14.4.19}$$

$$\left[\hat{\mathbf{H}}_0, \hat{\mathbf{K}}_2\right] = 2i\hat{\mathbf{K}}_1, \tag{14.4.20}$$

$$\left[\hat{\mathbf{K}}_1, \hat{\mathbf{K}}_2\right] = i\hat{\mathbf{K}}_2, \tag{14.4.21}$$

$$\left[\hat{\mathbf{K}}_1, \hat{\mathbf{G}}\right] = \frac{i}{2}\hat{\mathbf{G}}, \tag{14.4.22}$$

$$\left[\hat{\mathbf{K}}_2, \hat{\mathbf{G}}\right] = 0. \tag{14.4.23}$$

is satisfied. Then the operators (14.4.8)-(14.4.12) satisfy the Schrödinger algebra. However, the operators (14.4.8)-(14.4.12) are not hermitian and are not conserved. Then, the equivalence between the Black-Scholes and the free Schrödinger equation is not exactly, this last happen because the transformation (14.3.5) is not unitary.

Using other methods, the Black-Scholes-Merton symmetries have been studied in [Gazizov1988].

14.5 Summary

It was shown that the Black-Scholes equation is invariant under the Schrödinger group. In order to do this, the one dimensional free non-relativistic particle and its symmetries were revisited. To get the Black-Scholes equation symmetries, the particle mass was identified as the inverse of square of the volatility. Besides, using financial variables, a Schrödinger algebra representation was constructed. However, the operators of this last representation are not hermitian and not conserved.

Bibliografía

[Black1973] Black F. and Scholes M. (1973). **The pricing options and corporate liabilities,** Journal of Political Economy, vol 81: 637-659.

[Merton1973] Merton R. C. (1973). **Theory of Rational Option Pricing,** Bell J. Econ. and Management Sci, vol 4: 141-183.

[Mandelbrot1977] Mandelbrot B. B. (1997). **Fractals and scaling in finance**. Springer.

[Heston1993] Heston S. L. (1993). **A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options**. The Review of Financial Studies, vol 6: 327.

[Calvent2008] Calvet L. and Fisher A. (2008). **Multifractal Volatility: Theory, Forecasting, and Pricing**. Academic Press Advanced Finance, Elsevier Science.

[Tankov2003] Tankov P. (2003). **Financial Modelling with Jump Processes**. Springer.

[Carr2003] Carr P. and Wu L. (2003), Journal of Finance, vol 58: 753.

[Kleinert20015] Kleinert H. and Korbel J. (2007). **Option Pricing Beyond Black-Scholes Based on Double-Fractional Diffusion**. Preprint arXiv:1503.05655.

[Bouchaud2003] Bouchaud J. P and Potters M. (2003). **Theory of Financial Risk and Derivative Pricing, From Statical Physics to Risk Management**. Cambridge University Press.

[Baaquie2004] Baaquie B. E. (2004). **Quantum Finance,** Cambridge University Press.

[Maldacena2000] Aharony O., Gubser S. S., Maldacena J. and Ooguri H. (2000). **Large N field theories, string theory and gravity**. Phys. Rept, vol 323: 183.

[Hagen1972] Hagen C. R. (1972). **Scale and conformal transformations in galilean-covariant field theory**. Phys. Rev. D, vol 5: 377 .

[Niederer1972] Niederer U. (1972). **The maximal kinematical invariance group of the free Schrodinger equation**. Helv. Phys. Acta, vol 45: 802.

[Lie1882] **Lie S. (1882)**. Arch. Math. Nat. vid. (Kristiania), vol 6: 328.

[Romero2014] Romero J. M., Lavana U., Martínez E. (2014). **Schrödinger group and quantum finance**. International Journal of Pure Applied Mathematics, vol 90: 287.

[Gazizov1988] Gazizov R. K. and Ibragimov N. H. (1998). **Lie Symmetry Analysis of Differential Equations in Finance**. Nonlinear Dynamics, vol 17: 387-407.